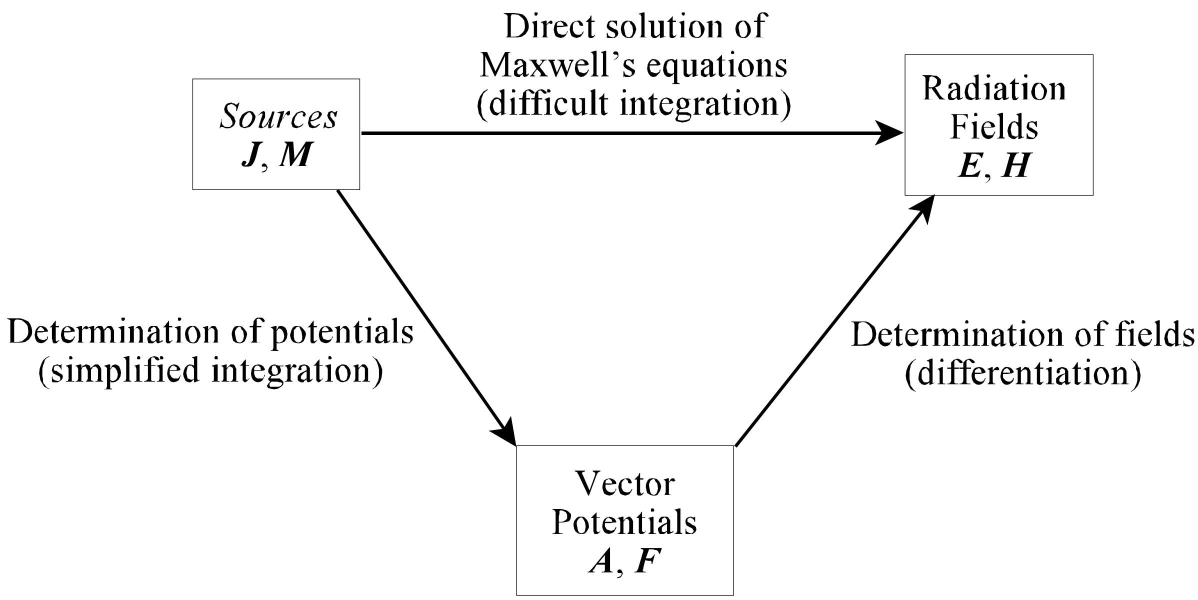
Determination of Antenna Radiation Fields Using Potential Functions

Sources of Antenna Radiation Fields → ***J*** - vector electric current density (A/m^2)
M - vector magnetic current density (V/m^2)

Some problems involving electric currents can be cast in equivalent forms involving magnetic currents (the use of magnetic currents is simply a mathematical tool, they have never been proven to exist).



A - magnetic vector potential (due to ***J***)
F - electric vector potential (due to ***M***)

In order to account for both electric current and/or magnetic current sources, the symmetric form of Maxwell's equations must be utilized to determine the resulting radiation fields. The symmetric form of Maxwell's equations include additional radiation sources (electric charge density - ρ and magnetic charge density ρ_m). However, these charges can always be related directly to the current via conservation of charge equations.

Maxwell's equations (symmetric, time-harmonic form)

$$\nabla \times \boldsymbol{E} = -j\omega \boldsymbol{B} - \boldsymbol{M} \qquad \text{(Faraday's law)}$$

$$\nabla \times \boldsymbol{H} = j\omega \boldsymbol{D} + \boldsymbol{J} \qquad \text{(Ampere's law)}$$

$$\nabla \cdot \boldsymbol{D} = \rho \qquad \text{(Gauss' law - electric fields)}$$

$$\nabla \cdot \boldsymbol{B} = \rho_m \qquad \text{(Gauss' law - magnetic fields)}$$

The use of potentials in the solution of radiation fields employs the concept of superposition of fields.

Electric current source ($\boldsymbol{J}$, ρ)	⇒	Magnetic vector potential ($\boldsymbol{A}$)	⇒	Radiation fields ($\boldsymbol{E}_A$, $\boldsymbol{H}_A$)
Magnetic current source ($\boldsymbol{M}$, ρ_m)	⇒	Electric vector potential ($\boldsymbol{F}$)	⇒	Radiation fields ($\boldsymbol{E}_F$, $\boldsymbol{H}_F$)

The total radiation fields ($\boldsymbol{E}$, $\boldsymbol{H}$) are the sum of the fields due to electric currents ($\boldsymbol{E}_A$, $\boldsymbol{H}_A$) and the fields due to the magnetic currents ($\boldsymbol{E}_F$, $\boldsymbol{H}_F$).

$$\boldsymbol{E} = \boldsymbol{E}_A + \boldsymbol{E}_F$$

$$\boldsymbol{H} = \boldsymbol{H}_A + \boldsymbol{H}_F$$

Maxwell's Equations (electric sources only ⇒ $\boldsymbol{F} = \boldsymbol{0}$)

$$\nabla \times \boldsymbol{E}_A = -j\omega \boldsymbol{B}_A \qquad (1a)$$

$$\nabla \times \boldsymbol{H}_A = j\omega \boldsymbol{D}_A + \boldsymbol{J} \qquad (1b)$$

$$\nabla \cdot \boldsymbol{D}_A = \rho \qquad (1c)$$

$$\nabla \cdot \boldsymbol{B}_A = 0 \qquad (1d)$$

Maxwell's Equations (magnetic sources only ⇒ $\boldsymbol{A} = \mathbf{0}$)

$$\nabla \times \boldsymbol{E_F} = -j\omega \boldsymbol{B_F} - \boldsymbol{M} \qquad (2a)$$

$$\nabla \times \boldsymbol{H_F} = j\omega \boldsymbol{D_F} \qquad (2b)$$

$$\nabla \cdot \boldsymbol{D_F} = 0 \qquad (2c)$$

$$\nabla \cdot \boldsymbol{B_F} = \rho_m \qquad (2d)$$

Based on the vector identity,

$$\nabla \cdot (\nabla \times \boldsymbol{G}) \equiv 0 \qquad \text{(for any vector } \boldsymbol{G}\text{)}$$

any vector with zero divergence (rotational or solenoidal field) can be expressed as the curl of some other vector. From Maxwell's equations with electric or magnetic sources only [Equations (1d) and (2c)], we find

$$\nabla \cdot \boldsymbol{B_A} = 0 \qquad \nabla \cdot \boldsymbol{D_F} = 0$$

so that we may define these vectors as

$$\boldsymbol{B_A} = \nabla \times \boldsymbol{A} \quad (3a) \qquad \boldsymbol{D_F} = -\nabla \times \boldsymbol{F} \quad (3b)$$

where $\boldsymbol{A}$ and $\boldsymbol{F}$ are the magnetic and electric vector potentials, respectively. The flux density definitions in Equations (3a) and (3b) lead to the following field definitions:

$$\boldsymbol{H_A} = \frac{1}{\mu} \nabla \times \boldsymbol{A} \quad (4a) \qquad \boldsymbol{E_F} = -\frac{1}{\epsilon} \nabla \times \boldsymbol{F} \quad (4b)$$

Inserting (3a) into (1a) and (3b) into (2b) yields

$$\nabla \times \boldsymbol{E_A} = -j\omega(\nabla \times \boldsymbol{A}) \quad (5a) \qquad \nabla \times \boldsymbol{H_F} = -j\omega(\nabla \times \boldsymbol{F}) \quad (5b)$$

Equations (5a) and (5b) can be rewritten as

$$\nabla \times \left[\boldsymbol{E}_A + j\omega \boldsymbol{A} \right] = 0 \qquad (6a) \qquad \nabla \times \left[\boldsymbol{H}_F + j\omega \boldsymbol{F} \right] = 0 \qquad (6b)$$

Based on the vector identity

$$\nabla \times (\nabla g) \equiv 0 \qquad \text{(for any scalar } g\text{)}$$

the bracketed terms in (6a) and (6b) represent non-solenoidal (lamellar or irrotational fields) and may each be written as the gradient of some scalar

$$\boldsymbol{E}_A + j\omega \boldsymbol{A} = -\nabla \phi_e \qquad (7a) \qquad \boldsymbol{H}_F + j\omega \boldsymbol{F} = -\nabla \phi_m \qquad (7b)$$

where ϕ_e is the electric scalar potential and ϕ_m is the magnetic scalar potential. Solving equations (7a) and (7b) for the electric and magnetic fields yields

$$\boldsymbol{E}_A = -j\omega \boldsymbol{A} - \nabla \phi_e \qquad (8a) \qquad \boldsymbol{H}_F = -j\omega \boldsymbol{F} - \nabla \phi_m \qquad (8b)$$

Equations (4a) and (8a) give the fields ($\boldsymbol{E}_A$, $\boldsymbol{H}_A$) due to electric sources while Equations (4b) and (8b) give the fields ($\boldsymbol{E}_F$, $\boldsymbol{H}_F$) due to magnetic sources. Note that these radiated fields are obtained by differentiating the respective vector and scalar potentials.

The integrals which define the vector and scalar potential can be found by first taking the curl of both sides of Equations (4a) and (4b):

$$\nabla \times \boldsymbol{H}_A = \frac{1}{\mu} \nabla \times \nabla \times \boldsymbol{A} \qquad (9a) \qquad \nabla \times \boldsymbol{E}_F = -\frac{1}{\epsilon} \nabla \times \nabla \times \boldsymbol{F} \qquad (9b)$$

According to the vector identity

$$\nabla \times \nabla \times \boldsymbol{G} = \nabla(\nabla \cdot \boldsymbol{G}) - \nabla^2 \boldsymbol{G} \qquad \text{(vector identity)}$$

and Equations (1b) and (2a), we find

$$\nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} = j\omega\mu\epsilon \boldsymbol{E}_{\boldsymbol{A}} + \mu \boldsymbol{J} \qquad (10a)$$

$$\nabla(\nabla \cdot \boldsymbol{F}) - \nabla^2 \boldsymbol{F} = j\omega\mu\epsilon \boldsymbol{H}_{\boldsymbol{F}} - \epsilon \boldsymbol{M} \qquad (10b)$$

Inserting Equations (7a) and (7b) into (10a) and (10b), respectively gives

$$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = \nabla(\nabla \cdot \boldsymbol{A}) + \nabla(j\omega\mu\epsilon\phi_e) - \mu \boldsymbol{J} \qquad (11a)$$

$$\nabla^2 \boldsymbol{F} + k^2 \boldsymbol{F} = \nabla(\nabla \cdot \boldsymbol{F}) + \nabla(j\omega\mu\epsilon\phi_m) - \epsilon \boldsymbol{M} \qquad (11b)$$

We have defined the rotational (curl) properties of the magnetic and electric vector potentials [Equations (3a) and (3b)] but have not yet defined the irrotational (divergence) properties. If we choose

$$\nabla \cdot \boldsymbol{A} = -j\omega\mu\epsilon\phi_e \qquad (12a)$$

$$\nabla \cdot \boldsymbol{F} = -j\omega\mu\epsilon\phi_m \qquad (12b)$$

Then, Equations (11a) and (11b) reduce to

$$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J} \qquad (13a)$$

$$\nabla^2 \boldsymbol{F} + k^2 \boldsymbol{F} = -\epsilon \boldsymbol{M} \qquad (13b)$$

The relationship chosen for the vector and scalar potentials defined in Equations (12a) and (12b) is defined as the *Lorentz gauge* [other choices for these relationships are possible]. Equations (13a) and (13b) are defined as inhomogenous Helmholtz vector wave equations which have solutions of the form

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu}{4\pi} \iiint_V \boldsymbol{J}(\boldsymbol{r}') \frac{e^{-jk|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|} dv' \qquad (14a)$$

$$\boldsymbol{F}(\boldsymbol{r}) = \frac{\epsilon}{4\pi} \iiint_V \boldsymbol{M}(\boldsymbol{r}') \frac{e^{-jk|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|} dv' \qquad (14b)$$

where $\boldsymbol{r}$ locates the field point (where the field is measured) and $\boldsymbol{r}'$ locates the source point (where the current is located). Similar inhomogeneous Helmholtz scalar wave equations can be found for the electric and magnetic scalar potentials.

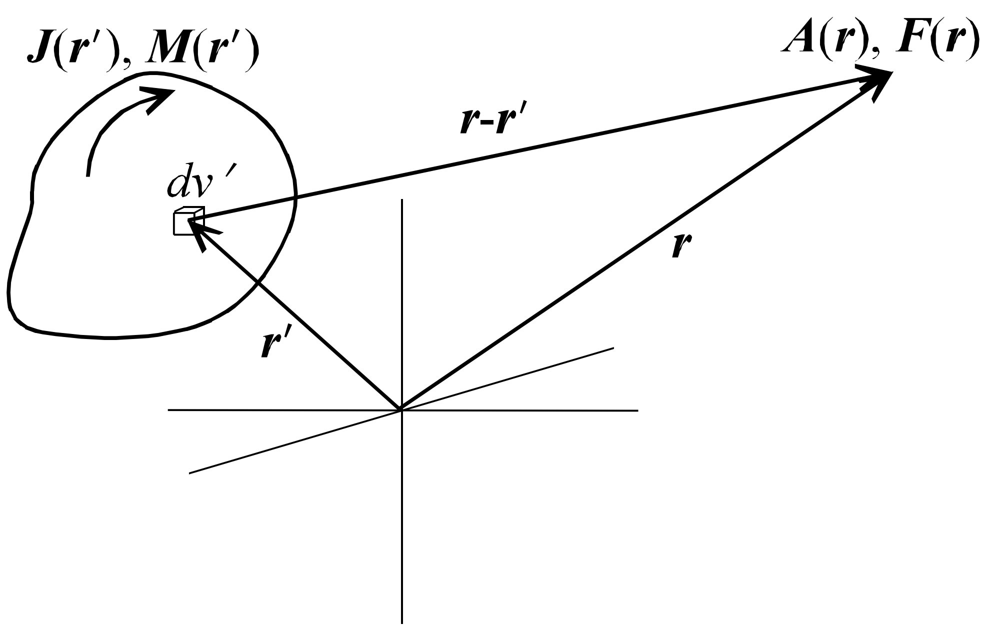
$$\nabla^2 \phi_e + k^2 \phi_e = -\frac{\rho}{\epsilon} \qquad (15a)$$

$$\nabla^2 \phi_m + k^2 \phi_m = -\frac{\rho_m}{\mu} \qquad (15b)$$

The solutions to the scalar potential equations are

$$\phi_e(\boldsymbol{r}) = \frac{1}{4\pi\epsilon} \iiint_V \rho(\boldsymbol{r}') \frac{e^{-jk|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|} dv' \qquad (16a)$$

$$\phi_m(\boldsymbol{r}) = \frac{1}{4\pi\mu} \iiint_V \rho_m(\boldsymbol{r}') \frac{e^{-jk|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|} dv' \qquad (16b)$$



Determination of Radiation Fields Using Potentials - Summary

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu}{4\pi} \iiint_V \boldsymbol{J}(\boldsymbol{r}') \frac{e^{-jk|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|} dv'$$

$$\begin{aligned} \boldsymbol{E}_{\boldsymbol{A}} &= -j\omega\boldsymbol{A} - \nabla\phi_e \\ &= -j\omega\boldsymbol{A} - \frac{j}{\omega\mu\epsilon}\nabla(\nabla\cdot\boldsymbol{A}) \\ &= -j\omega\left[\boldsymbol{A} + \frac{1}{k^2}\nabla(\nabla\cdot\boldsymbol{A})\right] \end{aligned}$$

$$\boldsymbol{H}_{\boldsymbol{A}} = \frac{1}{\mu}\nabla\times\boldsymbol{A}$$

$$\boldsymbol{F}(\boldsymbol{r}) = \frac{\epsilon}{4\pi} \iiint_V \boldsymbol{M}(\boldsymbol{r}') \frac{e^{-jk|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|} dv'$$

$$\begin{aligned} \boldsymbol{H}_{\boldsymbol{F}} &= -j\omega\boldsymbol{F} - \nabla\phi_m \\ &= -j\omega\boldsymbol{F} - \frac{j}{\omega\mu\epsilon}\nabla(\nabla\cdot\boldsymbol{F}) \\ &= -j\omega\left[\boldsymbol{F} + \frac{1}{k^2}\nabla(\nabla\cdot\boldsymbol{F})\right] \end{aligned}$$

$$\boldsymbol{E}_{\boldsymbol{F}} = -\frac{1}{\epsilon}\nabla\times\boldsymbol{F}$$

Notice in the previous set equations for the radiated fields in terms of potentials that the equations for $\boldsymbol{E}_A$ and $\boldsymbol{H}_F$ both contain a complex differentiation involving the gradient and divergence operators. In order to avoid this complex differentiation, we may alternatively determine $\boldsymbol{E}_A$ and $\boldsymbol{H}_F$ directly from Maxwell's equations once $\boldsymbol{E}_F$ and $\boldsymbol{H}_A$ have been determined using potentials. From Maxwell's equations for electric currents and magnetic currents, we have

$$\nabla \times \boldsymbol{H}_A = j\omega \boldsymbol{D}_A + \boldsymbol{J} \tag{1}$$

$$\nabla \times \boldsymbol{E}_F = -j\omega \boldsymbol{B}_F - \boldsymbol{M} \tag{2}$$

In antenna problems, the regions where we want to determine the radiated fields are away from the sources. Thus, we may set $\boldsymbol{J} = \boldsymbol{0}$ in Equation (1) to solve for $\boldsymbol{E}_A$ and set $\boldsymbol{M} = \boldsymbol{0}$ in Equation (2) to solve for $\boldsymbol{H}_F$. This yields

$$\boldsymbol{E}_A = \frac{1}{j\omega\epsilon} \nabla \times \boldsymbol{H}_A \qquad\qquad \boldsymbol{H}_F = -\frac{1}{j\omega\mu} \nabla \times \boldsymbol{E}_F$$

The total fields by superposition are

$$\boldsymbol{E} = \boldsymbol{E}_A + \boldsymbol{E}_F \qquad\qquad \boldsymbol{H} = \boldsymbol{H}_A + \boldsymbol{H}_F$$

which gives

$$\begin{aligned}\boldsymbol{E} = \boldsymbol{E}_A + \boldsymbol{E}_F &= -j\omega \boldsymbol{A} - \frac{j}{\omega\mu\epsilon} \nabla(\nabla \cdot \boldsymbol{A}) - \frac{1}{\epsilon} \nabla \times \boldsymbol{F} \\ &= \frac{1}{j\omega\epsilon} \nabla \times \boldsymbol{H}_A - \frac{1}{\epsilon} \nabla \times \boldsymbol{F}\end{aligned}$$

$$\begin{aligned}\boldsymbol{H} = \boldsymbol{H}_A + \boldsymbol{H}_F &= \frac{1}{\mu} \nabla \times \boldsymbol{A} - j\omega \boldsymbol{F} - \frac{j}{\omega\mu\epsilon} \nabla(\nabla \cdot \boldsymbol{F}) \\ &= \frac{1}{\mu} \nabla \times \boldsymbol{A} - \frac{1}{j\omega\mu} \nabla \times \boldsymbol{E}_F\end{aligned}$$

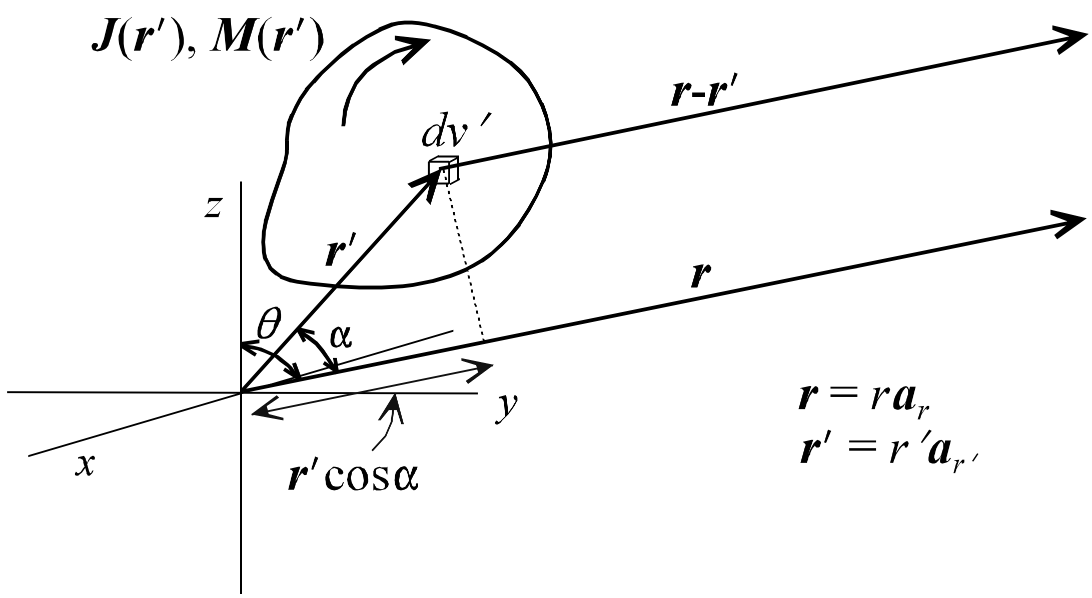
Antenna Far Fields in Terms of Potentials

As shown previously, the magnetic vector potential and electric vector potentials are defined as integrals of the (antenna) electric or magnetic current density.

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu}{4\pi}\iiint_V \boldsymbol{J}(\boldsymbol{r}')\,\frac{e^{-jk|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}\,dv'$$

$$\boldsymbol{F}(\boldsymbol{r}) = \frac{\epsilon}{4\pi}\iiint_V \boldsymbol{M}(\boldsymbol{r}')\,\frac{e^{-jk|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}\,dv'$$

If we are interested in determining the antenna far fields, then we must determine the potentials in the far field. We will find that the integrals defining the potentials simplify in the far field. In the far field, the vectors $\boldsymbol{r}$ and $\boldsymbol{r}-\boldsymbol{r}'$ becomes nearly parallel.



$$|\boldsymbol{r}-\boldsymbol{r}'| \approx r - r'\cos\alpha \qquad (1)$$

Using the approximation in (1) in the appropriate terms of the potential integrals yields

$$\frac{e^{-jk|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|} \approx \frac{e^{-jkr}e^{jkr'\cos\alpha}}{r - r'\cos\alpha} \tag{2}$$

If we assume that $r >> (r')_{max}$, then the denominator of (2) may be simplified to give

$$\frac{e^{-jk|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|} \approx \frac{e^{-jkr}e^{jkr'\cos\alpha}}{r} \tag{3}$$

Note that the r' term in the numerator complex exponential term in (3) cannot be neglected since it represents a phase shift term that may still be significant even in the far field. The *r*-dependent terms can be brought outside the integral since the potential integrals are integrated over the source (primed) coordinates. Thus, the far field integrals defining the potentials become

$$\boldsymbol{A}(\boldsymbol{r}) \approx \mu \frac{e^{-jkr}}{4\pi r} \iiint_V \boldsymbol{J}(\boldsymbol{r}')\, e^{jkr'\cos\alpha}\, dv' \tag{4}$$

$$\boldsymbol{F}(\boldsymbol{r}) \approx \epsilon \frac{e^{-jkr}}{4\pi r} \iiint_V \boldsymbol{M}(\boldsymbol{r}')\, e^{jkr'\cos\alpha}\, dv' \tag{5}$$

The potentials have the form of spherical waves as we would expect in the far field of the antenna. Also note that the complete *r*-dependence of the potentials is given outside the integrals. The r' term in the potential integrands can be expressed in terms of whatever coordinate system best fits the geometry of the source current. Spherical coordinates should always be used for the field coordinates in the far field based on the spherical symmetry of the far fields.

$$\boldsymbol{r}\cdot\boldsymbol{r}' = rr'\cos\alpha$$

$$r'\cos\alpha = \frac{\boldsymbol{r}\cdot\boldsymbol{r}'}{r} = \frac{xx' + yy' + zz'}{r}$$

Rectangular coordinate source

(field) $(x,y,z) = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$

(source) (x',y',z')

$$r'\cos\alpha = \frac{xx' + yy' + zz'}{r} = (x'\cos\phi + y'\sin\phi)\sin\theta + z'\cos\theta$$

Cylindrical coordinate source

(field) $(x,y,z) = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$

(source) $(x',y',z') = (\rho'\cos\phi', \rho'\sin\phi', z')$

$$r'\cos\alpha = \frac{xx' + yy' + zz'}{r} = \rho'\sin\theta\cos(\phi - \phi') + z'\cos\theta$$

Spherical coordinate source

(field) $(x,y,z) = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$

(source) $(x',y',z') = (r'\sin\theta'\cos\phi', r'\sin\theta'\sin\phi', r'\cos\theta')$

$$r'\cos\alpha = \frac{xx' + yy' + zz'}{r} = r'[\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')]$$

The results of the far field potential integrations in Equations (4) and (5) may be written as

$$\boldsymbol{A}(\boldsymbol{r}) \approx \frac{e^{-jkr}}{r}[A_r(\theta,\phi)\boldsymbol{a}_r + A_\theta(\theta,\phi)\boldsymbol{a}_\theta + A_\phi(\theta,\phi)\boldsymbol{a}_\phi]$$

$$\boldsymbol{F}(\boldsymbol{r}) \approx \frac{e^{-jkr}}{r}[F_r(\theta,\phi)\boldsymbol{a}_r + F_\theta(\theta,\phi)\boldsymbol{a}_\theta + F_\phi(\theta,\phi)\boldsymbol{a}_\phi]$$

The electric field due to an electric current source ($\boldsymbol{E}_A$) and the magnetic field due to a magnetic current source ($\boldsymbol{H}_F$) are defined by

$$\boldsymbol{E}_{\boldsymbol{A}} = -j\omega\boldsymbol{A} - \frac{j}{\omega\mu\epsilon}\nabla(\nabla\cdot\boldsymbol{A}) \tag{6}$$

$$\boldsymbol{H}_{\boldsymbol{F}} = -j\omega\boldsymbol{F} - \frac{j}{\omega\mu\epsilon}\nabla(\nabla\cdot\boldsymbol{F}) \tag{7}$$

If we expand the differential operators in Equations (6) and (7) in spherical coordinates, given the known r-dependence, we find that the $\boldsymbol{a}_r$-dependent terms cancel and all of the other terms produced by this differentiation are of dependence r^{-2} or lower. These field contributions are much smaller in the far field than the contributions from the first terms in Equations (6) and (7) which vary as r^{-1}. Thus, in the far field, $\boldsymbol{E}_A$ and $\boldsymbol{H}_F$ may be approximated as

$$\boldsymbol{E}_{\boldsymbol{A}} \approx -j\omega(A_\theta\,\boldsymbol{a}_\theta + A_\phi\,\boldsymbol{a}_\phi) \tag{8}$$

$$\boldsymbol{H}_{\boldsymbol{F}} \approx -j\omega(F_\theta\,\boldsymbol{a}_\theta + F_\phi\,\boldsymbol{a}_\phi) \tag{9}$$

The corresponding components of the fields ($\boldsymbol{H}_A$ and $\boldsymbol{E}_F$) can be found using the basic plane wave relationship between the electric and magnetic field in the far field of the antenna. Since the radiated far field must behave like a outward propagating spherical wave which looks essentially like a plane wave as $r\rightarrow\infty$, the far field components of $\boldsymbol{H}_A$ and $\boldsymbol{E}_F$ are related to the far field components of $\boldsymbol{E}_A$ and $\boldsymbol{H}_F$ by

$$E_{Ar} \approx 0 \qquad\qquad H_{Fr} \approx 0$$

$$E_{A\theta} \approx -j\omega A_\theta = \eta H_{A\phi} \qquad\qquad H_{F\phi} \approx -j\omega F_\phi = \frac{E_{F\theta}}{\eta}$$

$$E_{A\phi} \approx -j\omega A_\phi = -\eta H_{A\theta} \qquad\qquad H_{F\theta} \approx -j\omega F_\theta = -\frac{E_{F\phi}}{\eta}$$

Solving the previous equations for the individual components of $\boldsymbol{H}_A$ and $\boldsymbol{E}_F$ yields

$$H_{Ar} \approx 0 \qquad\qquad E_{Fr} \approx 0$$

$$H_{A\theta} \approx j\frac{\omega}{\eta}A_\phi = j\frac{k}{\mu}A_\phi \qquad\qquad E_{F\theta} \approx -j\omega\eta F_\phi = -j\frac{k}{\epsilon}F_\phi$$

$$H_{A\phi} \approx -j\frac{\omega}{\eta}A_\theta = -j\frac{k}{\mu}A_\theta \qquad\qquad E_{F\phi} \approx j\omega\eta F_\theta = j\frac{k}{\epsilon}F_\theta$$

Thus, once the far field potential integral is evaluated, the corresponding far field can be found using the simple algebraic formulas above (the differentiation has already been performed).

Duality

Duality - If the equations governing two different phenomena are identical in mathematical form, then the solutions also take on the same mathematical form (dual quantities).

Dual Equations

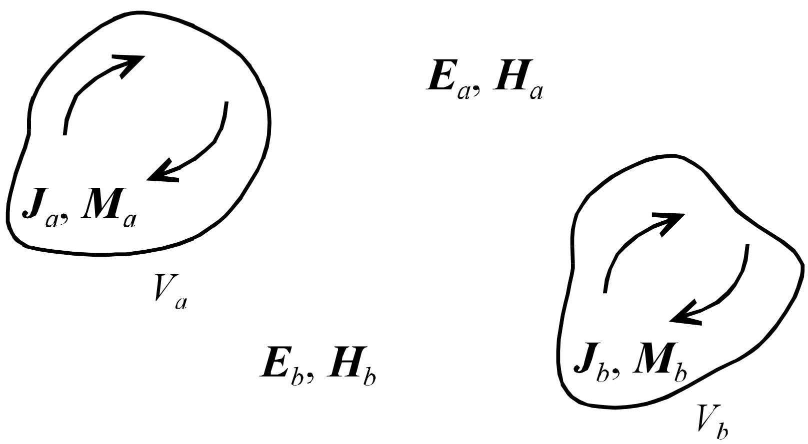
Electric Sources	Magnetic Sources
$\nabla \times \boldsymbol{H}_A = j\omega\epsilon \boldsymbol{E}_A + \boldsymbol{J}$	$-\nabla \times \boldsymbol{E}_F = j\omega\mu \boldsymbol{H}_F + \boldsymbol{M}$
$\nabla \times \boldsymbol{E}_A = -j\omega\mu \boldsymbol{H}_A$	$\nabla \times \boldsymbol{H}_F = j\omega\epsilon \boldsymbol{E}_F$
$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J}$	$\nabla^2 \boldsymbol{F} + k^2 \boldsymbol{F} = -\epsilon \boldsymbol{M}$
$\boldsymbol{A} = \dfrac{\mu}{4\pi} \iiint_V \boldsymbol{J} \dfrac{e^{-jk\lvert \boldsymbol{r} - \boldsymbol{r}' \rvert}}{\lvert \boldsymbol{r} - \boldsymbol{r}' \rvert} dv'$	$\boldsymbol{F} = \dfrac{\epsilon}{4\pi} \iiint_V \boldsymbol{M} \dfrac{e^{-jk\lvert \boldsymbol{r} - \boldsymbol{r}' \rvert}}{\lvert \boldsymbol{r} - \boldsymbol{r}' \rvert} dv'$
$\boldsymbol{H}_A = \dfrac{1}{\mu} \nabla \times \boldsymbol{A}$	$\boldsymbol{E}_F = -\dfrac{1}{\epsilon} \nabla \times \boldsymbol{F}$
$\boldsymbol{E}_A = -j\omega \boldsymbol{A} - \dfrac{j}{\omega\mu\epsilon} \nabla(\nabla \cdot \boldsymbol{A})$	$\boldsymbol{H}_F = -j\omega \boldsymbol{F} - \dfrac{j}{\omega\mu\epsilon} \nabla(\nabla \cdot \boldsymbol{F})$
$\nabla^2 \phi_e + k^2 \phi_e = -\dfrac{\rho}{\epsilon}$	$\nabla^2 \phi_m + k^2 \phi_m = -\dfrac{\rho_m}{\mu}$
$\phi_e = -\dfrac{1}{j\omega\mu\epsilon} \nabla \cdot \boldsymbol{A}$	$\phi_m = -\dfrac{1}{j\omega\mu\epsilon} \nabla \cdot \boldsymbol{F}$
$\phi_e = \dfrac{1}{4\pi\epsilon} \iiint_V \rho \dfrac{e^{-jk\lvert \boldsymbol{r} - \boldsymbol{r}' \rvert}}{\lvert \boldsymbol{r} - \boldsymbol{r}' \rvert} dv'$	$\phi_m = \dfrac{1}{4\pi\mu} \iiint_V \rho_m \dfrac{e^{-jk\lvert \boldsymbol{r} - \boldsymbol{r}' \rvert}}{\lvert \boldsymbol{r} - \boldsymbol{r}' \rvert} dv'$

Dual Quantities

Electric Sources	Magnetic Sources
$\boldsymbol{E}_{\boldsymbol{A}}$	$\boldsymbol{H}_{\boldsymbol{F}}$
$\boldsymbol{H}_{\boldsymbol{A}}$	$-\boldsymbol{E}_{\boldsymbol{F}}$
$\boldsymbol{J}$	$\boldsymbol{M}$
$\boldsymbol{A}$	$\boldsymbol{F}$
ρ	ρ_m
ϕ_e	ϕ_m
ϵ	μ
μ	ϵ
k	k
η	$1/\eta$
$1/\eta$	η

Reciprocity

Consider two sets of sources defined by ($\boldsymbol{J}_a$, $\boldsymbol{M}_a$) within the volume V_a and ($\boldsymbol{J}_b$, $\boldsymbol{M}_b$) within the volume V_b radiating at the same frequency. The sources ($\boldsymbol{J}_a$, $\boldsymbol{M}_a$) radiate the fields ($\boldsymbol{E}_a$, $\boldsymbol{H}_a$) while the sources ($\boldsymbol{J}_b$, $\boldsymbol{M}_b$) radiate the fields ($\boldsymbol{E}_b$, $\boldsymbol{H}_b$). The sources are assumed to be of finite extent and the region between the antennas is assumed to be isotropic and linear. We may write two separate sets of Maxwell's equations for the two sets of sources.



$$\nabla \times \boldsymbol{H}_a = j\omega\epsilon \boldsymbol{E}_a + \boldsymbol{J}_a \qquad (1a) \qquad\qquad \nabla \times \boldsymbol{H}_b = j\omega\epsilon \boldsymbol{E}_b + \boldsymbol{J}_b \qquad (1b)$$

$$-\nabla \times \boldsymbol{E}_a = j\omega\mu \boldsymbol{H}_a + \boldsymbol{M}_a \qquad (2a) \qquad\qquad -\nabla \times \boldsymbol{E}_b = j\omega\mu \boldsymbol{H}_b + \boldsymbol{M}_b \qquad (2b)$$

If we dot (1a) with $\boldsymbol{E}_b$ and dot (2b) with $\boldsymbol{H}_a$, we find

$$\boldsymbol{E}_b \cdot (\nabla \times \boldsymbol{H}_a) = j\omega\epsilon \boldsymbol{E}_a \cdot \boldsymbol{E}_b + \boldsymbol{J}_a \cdot \boldsymbol{E}_b \qquad (3a)$$

$$-\boldsymbol{H}_a \cdot (\nabla \times \boldsymbol{E}_b) = j\omega\mu \boldsymbol{H}_a \cdot \boldsymbol{H}_b + \boldsymbol{H}_a \cdot \boldsymbol{M}_b \qquad (3b)$$

Adding Equations (3a) and (3b) yields

$$\boldsymbol{E}_b \cdot (\nabla \times \boldsymbol{H}_a) - \boldsymbol{H}_a \cdot (\nabla \times \boldsymbol{E}_b)$$

$$= j\omega\epsilon \boldsymbol{E}_a \cdot \boldsymbol{E}_b + \boldsymbol{J}_a \cdot \boldsymbol{E}_b + j\omega\mu \boldsymbol{H}_a \cdot \boldsymbol{H}_b + \boldsymbol{H}_a \cdot \boldsymbol{M}_b$$

The previous equation may be rewritten using the following vector identity.

$$\nabla\cdot(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{B}\cdot(\nabla\times\boldsymbol{A}) - \boldsymbol{A}\cdot(\nabla\times\boldsymbol{B}) \qquad \text{(vector identity)}$$

which gives

$$-\nabla\cdot(\boldsymbol{E}_b\times\boldsymbol{H}_a) = j\omega\epsilon\boldsymbol{E}_a\cdot\boldsymbol{E}_b + \boldsymbol{J}_a\cdot\boldsymbol{E}_b + j\omega\mu\boldsymbol{H}_a\cdot\boldsymbol{H}_b + \boldsymbol{H}_a\cdot\boldsymbol{M}_b \qquad \text{(4a)}$$

If we dot (1b) with $\boldsymbol{E}_a$ and dot (2a) with $\boldsymbol{H}_b$, and perform the same operations, then we find

$$-\nabla\cdot(\boldsymbol{E}_a\times\boldsymbol{H}_b) = j\omega\epsilon\boldsymbol{E}_b\cdot\boldsymbol{E}_a + \boldsymbol{J}_b\cdot\boldsymbol{E}_a + j\omega\mu\boldsymbol{H}_b\cdot\boldsymbol{H}_a + \boldsymbol{H}_b\cdot\boldsymbol{M}_a \qquad \text{(4b)}$$

Subtracting (4a) from (4b) gives

$$-\nabla\cdot(\boldsymbol{E}_a\times\boldsymbol{H}_b - \boldsymbol{E}_b\times\boldsymbol{H}_a) = \boldsymbol{E}_a\cdot\boldsymbol{J}_b + \boldsymbol{H}_b\cdot\boldsymbol{M}_a - \boldsymbol{E}_b\cdot\boldsymbol{J}_a - \boldsymbol{H}_a\cdot\boldsymbol{M}_b \qquad \text{(5)}$$

If we integrate both sides of Equation (5) throughout all space and apply the divergence theorem to the left hand side, then

$$-\iint_S (\boldsymbol{E}_a\times\boldsymbol{H}_b - \boldsymbol{E}_b\times\boldsymbol{H}_a)\cdot d\boldsymbol{s} = \iiint_V (\boldsymbol{E}_a\cdot\boldsymbol{J}_b + \boldsymbol{H}_b\cdot\boldsymbol{M}_a - \boldsymbol{E}_b\cdot\boldsymbol{J}_a - \boldsymbol{H}_a\cdot\boldsymbol{M}_b)\,dv \qquad \text{(6)}$$

The surface on the left hand side of Equation (6) is a sphere of infinite radius on which the radiated fields approach zero. The volume V includes all space. Therefore, we may write

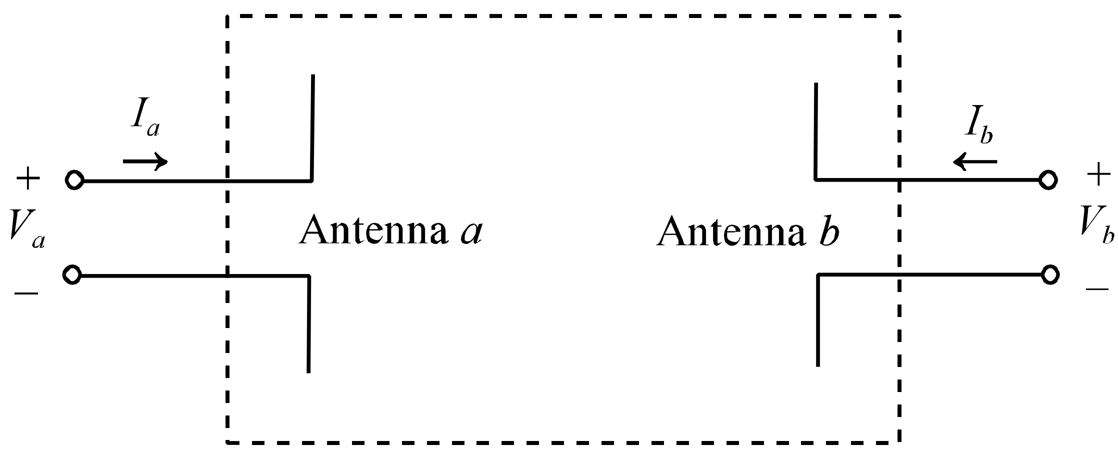
$$\iiint_V (\boldsymbol{E}_a\cdot\boldsymbol{J}_b - \boldsymbol{H}_a\cdot\boldsymbol{M}_b)\,dv = \iiint_V (\boldsymbol{E}_b\cdot\boldsymbol{J}_a - \boldsymbol{H}_b\cdot\boldsymbol{M}_a)\,dv \qquad \text{(7)}$$

Note that the left hand side of the previous integral depends on the "*b*" set of sources while the right hand side depends on the "*a*" set of sources. Since we have limited the sources to the volumes V_a and V_b, we may limit the volume integrals in (7) to the respective source volumes so that

$$\iiint_{V_b} (\boldsymbol{E}_a \cdot \boldsymbol{J}_b - \boldsymbol{H}_a \cdot \boldsymbol{M}_b)\, dv = \iiint_{V_a} (\boldsymbol{E}_b \cdot \boldsymbol{J}_a - \boldsymbol{H}_b \cdot \boldsymbol{M}_a)\, dv \qquad (8)$$

Equation (8) represents the general form of the reciprocity theorem.

We may use the reciprocity theorem to analyze a transmitting-receiving antenna system. Consider the antenna system shown below. For mathematical simplicity, let's assume that the antennas are perfectly-conducting, electrically short dipole antennas.



The source integrals in the general 3-D reciprocity theorem of Equation (8) simplify to line integrals for the case of wire antennas.

$$\int_{L_a} \boldsymbol{E}_b \cdot \boldsymbol{I}_a \, dl' = \int_{L_b} \boldsymbol{E}_a \cdot \boldsymbol{I}_b \, dl'$$

Furthermore, the electric field along the perfectly conducting wire is zero so that the integration can be reduced to the antenna terminals (gaps).

$$\int_{g_a} \boldsymbol{E_b} \cdot \boldsymbol{I_a}\, dl' = \int_{g_b} \boldsymbol{E_a} \cdot \boldsymbol{I_b}\, dl'$$

If we further assume that the antenna current is uniform over the electrically short dipole antennas, then

$$I_a \int_{g_a} \boldsymbol{E_b} \cdot \boldsymbol{dl}' = I_b \int_{g_b} \boldsymbol{E_a} \cdot \boldsymbol{dl}'$$

The line integral of the electric field transmitted by the opposite antenna over the antenna terminal gives the resulting induced open circuit voltage.

$$I_a V_a = I_b V_b$$

If we write the two port equations for the antenna system, we find

$$V_a = Z_{aa} I_a + Z_{ab} I_b$$

$$V_b = Z_{ba} I_a + Z_{bb} I_b$$

Note that the impedances Z_{ab} and Z_{ba} have been shown to be equal from the reciprocity theorem.

$$Z_{ba} = \left. \frac{V_b}{I_a} \right|_{I_b = 0}$$

$$Z_{ab} = \left. \frac{V_a}{I_b} \right|_{I_a = 0}$$

$$Z_{ab} = \frac{V_a}{I_b} = \frac{V_b}{I_a} = Z_{ba}$$

Therefore, if we place a current source on antenna *a* and measure the response at antenna *b*, then switch the current source to antenna *b* and measure the response at antenna *a*, we find the same response (magnitude and phase). Also, since the transfer impedances (Z_{ab} and Z_{ba}) are identical, the transmit and receive patterns of a given antenna are identical. Thus, we may measure the pattern of a given antenna in either the transmitting mode or receiving mode, whichever is more convenient.